Solutions

5.2: Solving Linear Programming Problems Graphically

Linear Programming (LP) Problems. A linear programming problem in two unknowns x and y is one in which we are to find the maximum or minimum value of a linear expression

$$ax + by$$

called the objective function, subject to a number of linear constraints of the form

$$cx + dy \leq e \text{ or } cx + dy \geq e.$$

The largest or smallest value of the objective function is called the optimal value, and a pair of values x and y that gives the optimal value constitutes an optimal solution.

Example 1. Maximize the function $p = x + y$ subject to the constraints

$$x + 2y \leq 12, \quad 2x + y \leq 12, \quad x \geq 0, \quad y \geq 0.$$

$x + 2y \leq 12 \Rightarrow y \leq \frac{12-x}{2}$

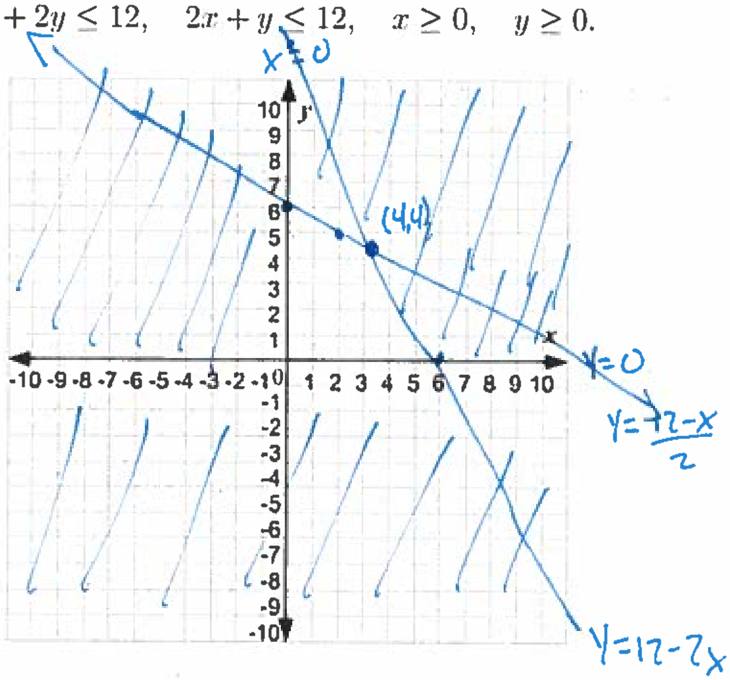
$2x + y \leq 12 \Rightarrow y \leq 12 - 2x$

$x + 2y = 12$

$2(2x + y = 12)$

$-3x = -12$

$x = 4, y = 4$



Corners at $(0,0), (0,6), (4,4), (6,0)$:

Which corner maximizes $p = x + y$?

$p(0,0) = 0 + 0 = 0$

$p(0,6) = 0 + 6 = 6$

$p(4,4) = 4 + 4 = 8$

$p(6,0) = 6 + 0 = 6$

The optimal value is 8 with optimal solution $(4,4)$.

Fundamental Theorem of Linear Programming.

- If an LP problem has optimal solutions, then at least one of these solutions occurs at a corner point of the feasible region.
- LP problems with bounded, nonempty feasible regions always have optimal solutions.

Example 2. Acme Baby Foods mixes two strengths of apple juice. One quart of Beginner's juice is made from 30 fluid ounces of water and 2 fluid ounces of apple juice concentrate. One quart of Advanced juice is made from 20 fluid ounces of water and 12 fluid ounces of apple juice concentrate. Every day Acme has available 30,000 fluid ounces of water and 3,600 fluid ounces of concentrate. Acme makes a profit of 20¢on each quart of Beginner's juice and 30¢on each quart of Advanced juice. How many quarts of each should Acme make each day to get the largest profit? How would this change if Acme made a profit of 40¢on Beginner's juice and 20¢on Advanced juice?

	Beginner's (x)	Advanced (y)	Total
Water	30	20	30,000
juice	2	12	3,600

Profit $= p = 20x + 30y$ w/ constraints $30x + 20y \le 30{,}000$

$2x + 12y \le 3600$

Implicit Constraints : $x \ge 0, y \ge 0$

$30x + 20y = 30{,}000 \Rightarrow y = \frac{3000 - 3x}{2}$

$2x + 12y = 3{,}600 \Rightarrow y = \frac{1800 - x}{6}$

Corners at $(0,0), (1000,0), (900,150), (0,3$

$p(0,0) = 0$ $\quad$ $p(900,150) = 22{,}500$

$p(1000,0) = 20{,}000$ $\quad$ $p(0,300) = 9{,}000$

Optimal value $= 22{,}500$ ¢ $= \$225$

Optimal Solution $= (900,150)$

If $p = 40x + 20y$, then

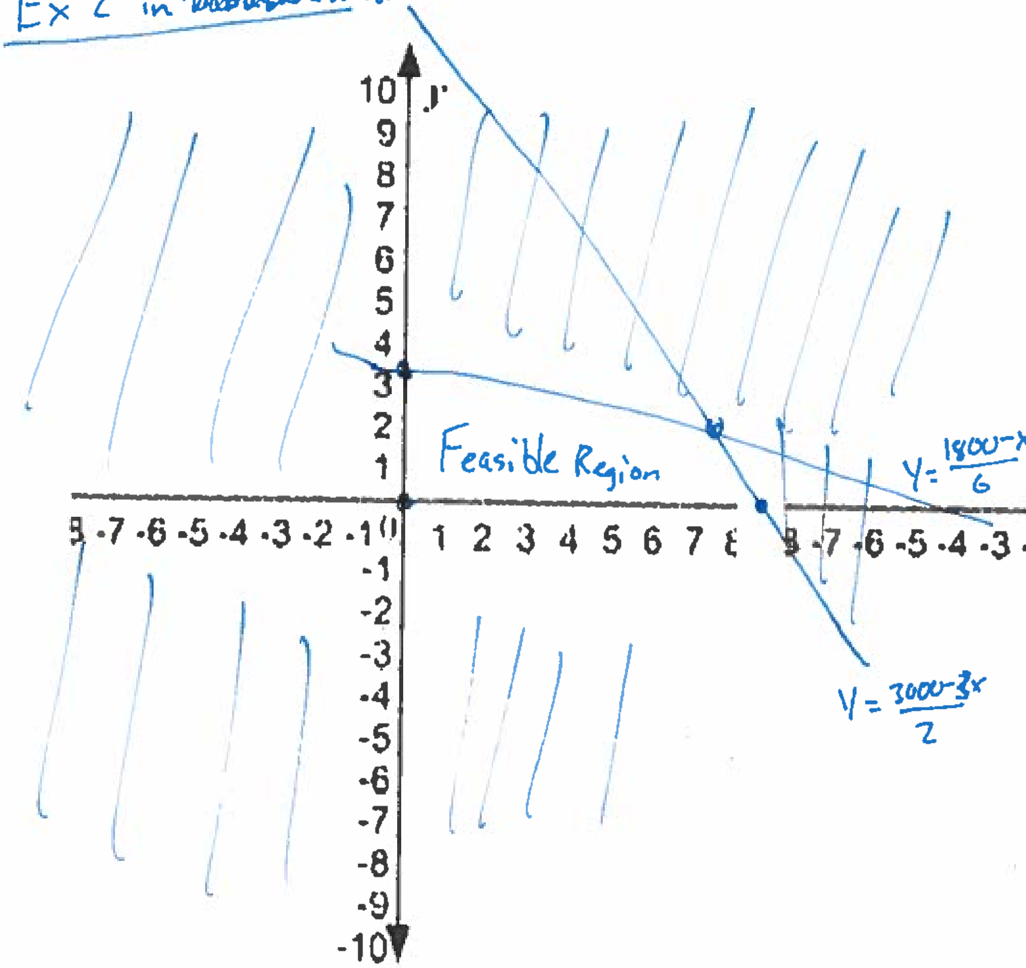
$p(0,0) = 0$ $\quad$ $p(900,150) = 39{,}000$

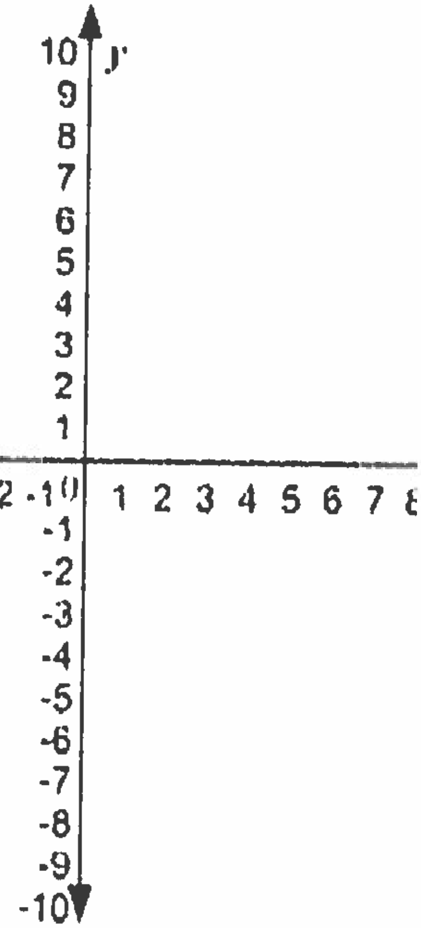
$p(1000,0) = 40{,}000$ $\quad$ $p(0,300) = 6{,}000$

$\Rightarrow$ Optimal value $= 40{,}000$ ¢ $= \$400$

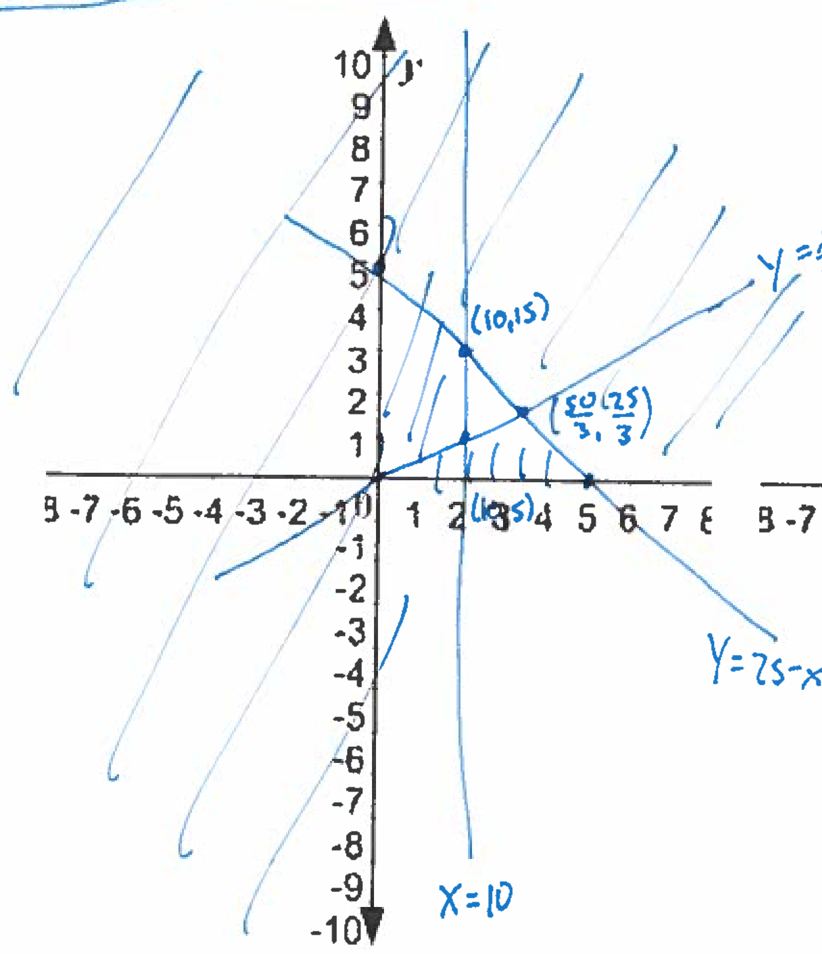
Optimal Solution $= (1000,0)$

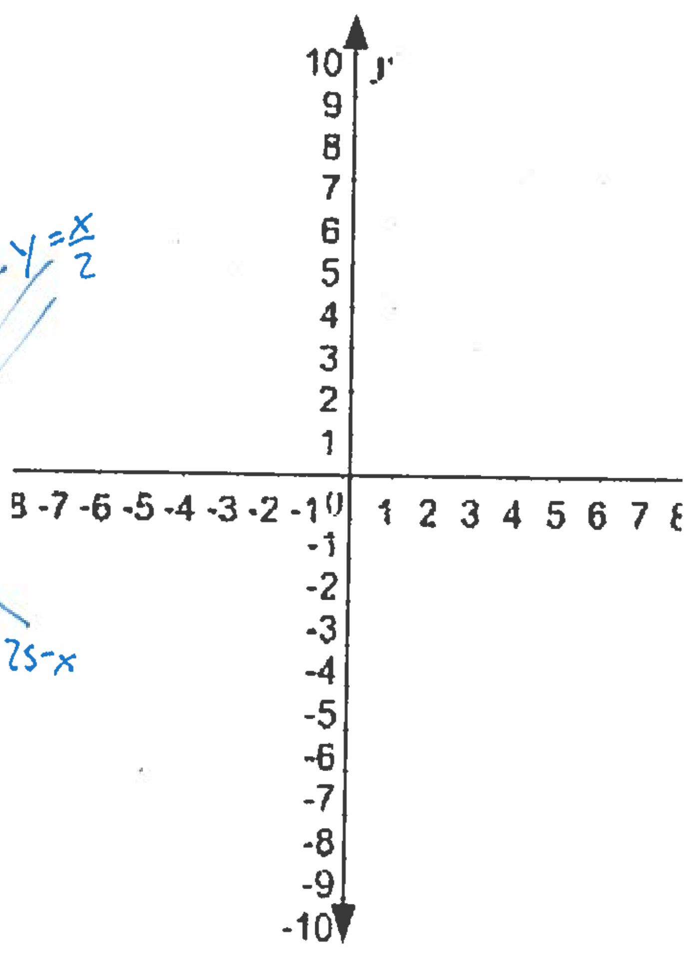
Ex 2 in ~~thousands~~ hundreds





Ex 3 (each tick is 5 million)





Example 3. The Solid Trust Savings & Loan Company has set aside \$ 25 million for loans to home buyers. Its policy is to allocate at least \$ 10 million annually for luxury condominiums. A government housing development grant it receives requires, however, that at least one third of its total loans be allocated to low-income housing.

(a) Solid Trust's return on condominiums is 12% and its return on low-income housing is 10%. How much should the company allocate for each type of housing to maximize its total return?

(b) Redo part (a), assuming that the return is 12% on both condominiums and low-income housing.

Constraints: (in millioms of \$)

Luxury Condiminiums $= x$

Low-income Housing $= y$

$x \geq 10$

$x+y \leq 25 \Rightarrow y \leq 25-x$

$y \geq \frac{1}{3}(x+y) \Rightarrow y \geq \frac{x}{2}$

Corners at $(10,15)$, $(50/3, 25/3)$, $(10,5)$

(a) Return $= r = .12x + .1y$. So $r(10,15) = 2.7$, $r(\frac{50}{3}, \frac{25}{3}) = 2.833$, $r(10,5) = 1.7$

optimal value $=$ \$2.833 million w/ optimal solution $(\frac{50}{3}, \frac{25}{3})$

(b) Return $= r = .12x + .12y$. So $r(10,15) = 3$, $r(\frac{50}{3}, \frac{25}{3}) = 3$, $r(10,5) = 1.8$

Optimal value $=$ \$3 million w/ optimal solutions $(10,15)$, $(\frac{50}{3}, \frac{25}{3})$ and everything in between.

(For instance $(12,13)$).

Example 4. Solving LP problems in two unknowns with unbounded feasible regions. You are the manager of a small store that specializes in hats, sunglasses, and other accessories. You are considering a sales promotion of a new line of hats and sunglasses. You will offer the sunglasses only to those who purchase two or more hats, so you will sell at least twice as many hats as sunglasses. Moreover, your supplier tells you that, due to seasonal demand, your order of sunglasses cannot exceed 100 pairs. To ensure that the salte items fill out the large display you have set aside, you estimate that you should order at least 210 items in all.

(a) Assume that you will lose \$ 3 on every hat and \$ 2 on every pair of sunglasses sold. Given the constraints above, how many hats and sunglasses should you order to lose the least amount of money in the sales promotion?

(b) Suppose instead that you lose \$ 1 on every hat, but make a profit of \$ 5 on every pair of sunglasses. How many hats and sunglasses should you order to lose the least amount of money in the sales promotion?

(c) Now suppose that you make a profit of \$ 1 on every hat, but lose \$ 5 on every pair of sunglasses. How many hats and sunglasses should you order to lose the least amount of money in the sales promotion?

Hats = x

Sunglasses = y

Constraints: $x \geq 0,\ y \geq 0,\ 2y \leq x,\ y \leq 100,\ x+y \geq 210$

$\Downarrow$

$y \geq 210 - x$

Corners at $(140,70)$, $(200,100)$ and $(210,0)$

w/ two additional artificial boundary corners $(300,100)$ and $(300,0)$.

(a) Objective function: $\text{Cost} = C = 3x + 2y$

$C(140,70) = 560$

$C(200,100) = 800$

$C(210,0) = 630$

$C(300,100) = 1{,}100$

$C(300,0) = 900$

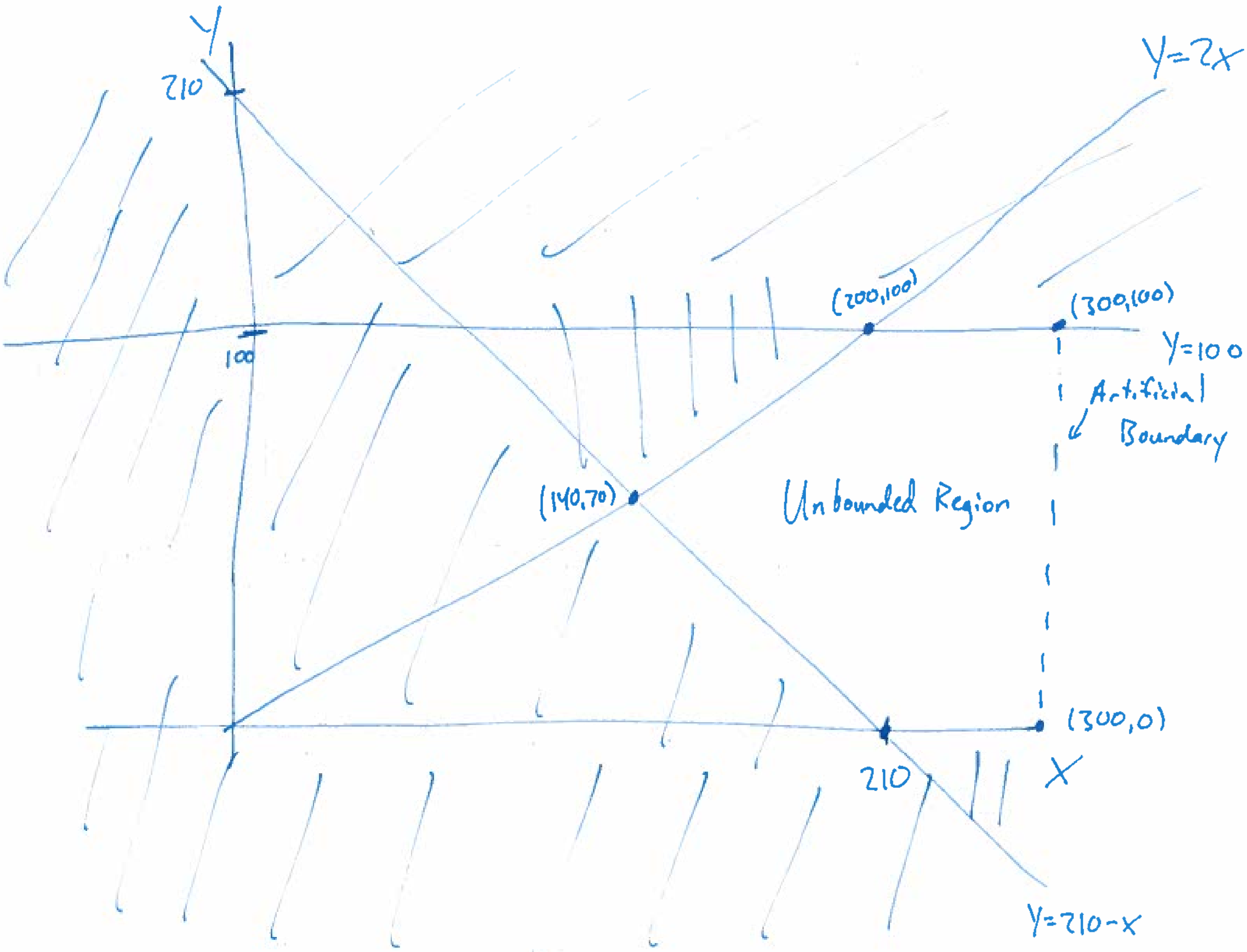
minimized at $(140,70)$ with \$560 lost.

(c) ~~Cost~~ Profit = p = $x - 5y$

~~maximized~~ ~~Minimized~~ maximized at ~~(300,0)~~ $(300,0)$ w/ \$300 ~~lost~~ gained. Since this is on the artifial boundary the cost cannot be minimized in the unbounded region.

(b) $\text{Cost} = C = -x + 5y$

maximized at $(200,100)$ w/ \$300 gained.



Y
210
Y=2X
(200,100)
(300,100)
100
Y=100
Artificial
Boundary
(140,70)
Unbounded Region
(300,0)
210
X
Y=210-X

AFL Entropy

$A = C^*$-alg., V = Bdd, ~~s.a.~~ lin. fcnals on A.

$K = \{w \in V : w(a^*a) \geq 0\}$, $\theta : A \to A$ an isometric automorphism (so ~~trace~~ preserving K,))

and w is an invariant state; i.e. $w \circ \theta = w$.

Let $X = (x_1, \dots, x_k)$ be an operational part. of unity.

Define $T : V \to V$ by $T(v)(a) = v(\theta(a))$

and $\gamma(\{i\})(v)(a) = v(x_i^* a x_i)$ st $\gamma_j(\{i\})(v) := v(x_i^* x_i)$

Then

$u^w_{0,\dots,n-1}(i_0, \dots, i_{n-1}) = \tau(\theta^{n-1} \circ \gamma(i_{n-1}) \circ \theta^{n-2} \circ \cdots \circ \theta(i_1) \circ \gamma(i_1) \circ \gamma(i_0))\, w)$

flip them again !!

$\#\{a\}$

$= \tau((\gamma_{n-1}(i_{n-1}) \circ \cdots \circ \gamma_0(i_0))w)$, where $\gamma_n(E) = \theta^{-n} \circ \gamma(E) \circ \theta^n$

$= \tau((\theta^{-(n-1)} \circ \gamma(i_{n-1}) \circ \theta \circ \gamma(i_{n-2}) \circ \cdots \circ \theta \circ \gamma(i_0))w)$

$= \tau((\cdots \theta)\, w(x_{i_0}^* a x_{i_0}))$ $w \circ \theta = w \Rightarrow \theta \circ \rho = \rho$

$= \tau((\cdots \circ \gamma(i_1))\, w(\theta(x_{i_0}^*) \overset{\theta}{a} (x_{i_0})))$

$= \tau(w(\theta^{-(n-1)}(w))((x_{i_{n-1}} \theta(x_{i_{n-2}} \theta(\cdots \theta(x_{i_0}))))^* a (\dots)))$

?? It is more natural above. Recall T-F paper?

$= \tau(w(x_{i_0}^* \theta(x_{i_1}^*) \cdots \theta^{n-1}(x_{i_{n-1}}^*) a\, \theta^{n-1}(x_{i_{n-1}}) \cdots x_{i_0})) \overset{?}{=} \sum_i w(x_i^* x_i)$